① $\int x^2(x^3+3)^3\,dx$; $u = x^3+3$

$\frac{du}{dx} = 3x^2$

$= \int \cancel{x^2}^{1}\, u^3 \,\frac{du}{3\cancel{x^2}_{1}}$

$= \frac{1}{3}\int u^3\,du = \frac{1}{3}\cdot\frac{u^4}{4} + C$

$= \frac{1}{12}(x^3+3)^4 + C$

② $\int (5x^4+4x)(x^5+2x^2+1)^3\,dx$

$= \int \cancel{(5x^4+4x)}^{1}\, u^3 \,\frac{du}{\cancel{5x^4+4x}}$, 1 $= \int u^3\,du = \frac{u^4}{4} + C$

$= \frac{(x^5+2x^2+1)^4}{4} + C$

③ $\int \frac{3x-2}{(3x^2-4x-1)^3}\,dx$; $u = 3x^2-4x-1$

$= \int \frac{\cancel{3x-2}^{1}}{u^3}\,\frac{du}{\cancel{6x-4}\ 2} = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C$

$= -\frac{1}{4}(3x^2-4x-1)^{-2} + C$

④ $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$; let $u = x^3+1$

$= \int \frac{\cancel{x^2}^{1}}{\sqrt{u}}\,\frac{du}{3\cancel{x^2}_{1}} = \frac{1}{3}\int u^{-\frac{1}{2}}\,du = \frac{1}{3}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)$

$= \frac{2}{3}\sqrt{x^3+1} + C$

⑤ $\displaystyle\int \frac{21x^2+14}{\sqrt{x^3+2x+1}}\,dx$; $u = x^3+2x+1$

$$= \int \frac{7(\cancel{3x^2+2})^{1}}{\sqrt{u}}\;\frac{du}{\cancel{3x^2+2}_{1}} = 7\int u^{-\frac{1}{2}}du = \frac{7\cdot u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 14\sqrt{x^3+2x+1} + c$$

⑥ $\displaystyle\int \frac{x+5}{\sqrt{1-2x^2}}\,dx$

$$= \int \frac{x}{\sqrt{1-2x^2}}\,dx + \int \frac{5}{\sqrt{1-2x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{u}}\;\frac{du}{-4x} + 5\int \overbrace{\frac{1}{\sqrt{1-(\sqrt{2}x)^2}}}^{\rightarrow\, u=\sqrt{2}x}\,dx$$

$$= -\frac{1}{4}\int u^{-\frac{1}{2}}du + 5\;\frac{\sin^{-1}\left(\frac{\sqrt{2}x}{1}\right)}{\sqrt{2}} + C$$

$$\int u^{-\frac{1}{2}}du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{u} + c$$

divide by coefficient of x

$$= -\frac{1}{4}(2)\sqrt{1-2x^2} + \frac{5}{\sqrt{2}}\sin^{-1}(\sqrt{2}x) + c$$

$$= -\frac{1}{2}\sqrt{1-2x^2} + \frac{5}{\sqrt{2}}\sin^{-1}(\sqrt{2}x) + c$$

⑦

$$\int \left(\frac{x}{\sqrt{3x^4-1}} \right)^3 dx$$

$$= \int \frac{x^3}{(3x^4-1)^{\frac{3}{2}}} dx \quad ; u = 3x^4 - 1$$

$$= \int x^3 (u)^{-\frac{3}{2}} \frac{du}{12x^3} = \frac{1}{12} \int u^{-\frac{3}{2}} du = \frac{1}{12}\left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \right) + C$$

$$= - \frac{1}{6\sqrt{3x^4-1}} + C$$

✓

⑧

$$\int \frac{x^3}{x^4+3} dx \quad ; u = x^4 + 3$$

$$= \int \frac{\overset{1}{\cancel{x^3}}}{u} \frac{du}{4\underset{1}{\cancel{x^3}}}$$

$$= \frac{1}{4} \ln|x^4+3| + C$$

✓

⑨ $$\int \frac{2x+1}{2x^2+2x+5}\,dx$$

$$= \int \frac{\overset{1}{\cancel{2x+1}}}{u}\,\frac{du}{\underset{2}{\cancel{4x+2}}}$$

$$= \frac{1}{2}\ln\left|2x^2+2x+5\right| + C$$

⑩ $$\int \frac{x}{\sqrt{1-4x-x^2}}\,dx = \int \frac{A(-4-2x)+B}{\sqrt{1-4x-x^2}}\,dx$$

$x = A(-4-2x)+B$

$1 = -2A$, $-4A+B=0$

$A = -\frac{1}{2}$, $B = -2$

$1-4x-x^2$

$= 1-(x^2+4x)$

$= 1-[(x+2)^2-2^2]$

$= \sqrt{5}^2-(x+2)^2$

$$= -\frac{1}{2}\underbrace{\int \frac{-4-2x}{\sqrt{1-4x-x^2}}\,dx}_{\text{cancel.}} - 2\underbrace{\int \frac{1}{\sqrt{1-4x-x^2}}\,dx}_{\text{MF27}}$$

$$= -\frac{1}{2}\int \frac{\overset{1}{\cancel{-4-2x}}}{\sqrt{u}}\,\frac{du}{\underset{1}{\cancel{-4-2x}}} - 2\int \frac{1}{\sqrt{\sqrt{5}^2-(x+2)^2}}\,dx$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du - 2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

$$= -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) - 2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

$$= -\sqrt{1-4x-x^2} - 2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

⑪

$$\int \frac{x+4}{x^2+x+1}\,dx = \int \frac{A(2x+1)+B}{x^2+x+1}\,dx \;;$$

$x+4 = A(2x+1)+B$

$2A=1\,,\quad A+B=4$

$A=\frac{1}{2}\qquad B=4-\frac{1}{2}=\frac{7}{2}$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{7}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{7}{2}\int \underbrace{\frac{1}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}}_{\text{MF27}}\,dx$$

$$= \underline{\ln\left|x^2+x+1\right|^{\frac{1}{2}} + \frac{7}{2}\left(\frac{1}{\frac{\sqrt{3}}{2}}\right)\tan^{-1}\left(\frac{2(x+\frac{1}{2})}{2(\frac{\sqrt{3}}{2})}\right)} + c$$

$$= \ln\sqrt{x^2+x+1} + \frac{7}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

⑫

$$\int \frac{2x+1}{x^2+x+1}\,dx \;;\; u = x^2+x+1$$

$$= \int \frac{\overset{1}{\cancel{2x+1}}}{u}\,\frac{du}{\underset{1}{\cancel{2x+1}}}$$

$$= \ln\left|x^2+x+1\right| + C$$

Method I $(u = \sqrt{x} + 1)$

⑬ $$\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^9}\,dx$$

$u = \sqrt{x} + 1$

$\frac{du}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$$= \int \frac{1}{\cancel{\sqrt{x}}}\cdot\frac{1}{u^9}\,\frac{du}{\frac{1}{2}\cdot\frac{1}{\cancel{\sqrt{x}}}}$$

$$= 2\int u^{-9}\,du = \frac{2u^{-8}}{-8} + C = -\frac{1}{4}u^{-8} + C$$

$$= -\frac{1}{4(\sqrt{x}+1)^8} + C$$

Method II $(u = \sqrt{x})$

$$\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^9}\,dx \text{ ; let } u = \sqrt{x}$$

$$= \int \frac{1}{\cancel{\sqrt{x}}}^{1}\cdot\frac{1}{(u+1)^9}\,\frac{du}{\frac{1}{2}\cdot\frac{1}{\cancel{\sqrt{x}}}_{1}}$$

$$= 2\int (u+1)^{-9}\,du$$

$$= 2\,\frac{(u+1)^{-8}}{(-8)} + C$$

$$= -\frac{1}{4(\sqrt{x}+1)^8} + C$$

A.

⑭

Method I ($u = \sqrt{x}$)

$$\int \frac{1}{\sqrt{x}} \cdot \frac{1}{e^{\sqrt{x}}} \, dx$$

$$= \int \frac{1}{\cancel{\sqrt{x}}} \cdot \frac{1}{e^{u}} \, \frac{du}{\frac{1}{2} \cdot \frac{1}{\cancel{\sqrt{x}}}}$$

$$= 2\int e^{-u} \, du = \frac{2e^{-u}}{-1} + C$$

$$= -2e^{-\sqrt{x}} + C$$

Method II ($u = e^{\sqrt{x}}$)

$$\int \frac{1}{\cancel{\sqrt{x}}} \cdot \frac{1}{u} \, \frac{du}{e^{\sqrt{x}} \cdot \frac{1}{2} \cdot \frac{1}{\cancel{\sqrt{x}}}}$$

$$= 2\int \frac{1}{u e^{\sqrt{x}}} \, du = 2\int \frac{1}{u \cdot u} \, du = 2\int u^{-2} \, du$$

$$= \frac{2u^{-1}}{-1} + C$$

$$= -2e^{-\sqrt{x}} + C$$

⑮ $\int \frac{(\ln x)^8}{x}\,dx$; $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$= \int \cancel{\frac{1}{x}}\, u^8 \,\frac{du}{\cancel{\frac{1}{x}}}$

$= \frac{u^9}{9} + C$

$= \frac{(\ln x)^9}{9} + C$

⑯ $\int \frac{1}{x \ln x^3}\,dx$

$= \frac{1}{3}\int \frac{1}{x \ln x}\,dx$

$= \frac{1}{3}\int \cancel{\frac{1}{x}} \cdot \frac{1}{u}\,\frac{du}{\cancel{\frac{1}{x}}}$

$= \frac{1}{3}\ln\left|\ln x\right| + C$

vs $\int \frac{1}{x(\ln x)^3}\,dx$

$(\ln x)^3 = (\ln x)(\ln x)(\ln x)$

$\ln(x^3) = 3\ln x$

$= \int \cancel{\frac{1}{x}}\,\frac{1}{u^3}\,\frac{du}{\cancel{\frac{1}{x}}}$

$= \int u^{-3}\,du = \frac{u^{-2}}{-2} + C$

$= -\frac{1}{2(\ln x)^2} + C$.

⑰ $\int x e^{x^2} e^{x^2-1} \, dx$

$= \int x \, e^{x^2+x^2-1} \, dx$

$= \int x \, e^{2x^2-1} \, dx$; $u = \underline{2x^2-1}$

$= \int \overset{1}{\cancel{x}} \, e^{u} \, \frac{du}{4\underset{1}{\cancel{x}}} = \frac{1}{4} e^{u} + c$

$= \frac{1}{4} e^{2x^2-1} + c$.

⑱ $\int \frac{e^x (1)}{e^x(1+e^{-x})} \, dx$

Note: $e^x \cdot e^{-x} = e^{x-x} = e^0 = 1$

$\hookrightarrow = e^x \cdot \frac{1}{e^x} \nearrow$

$= \int \frac{e^x}{e^x + e^x \cdot e^{-x}} \, dx$

$= \int \frac{e^x}{e^x+1} \, dx$

$= \int \frac{\overset{1}{\cancel{e^x}}}{u} \, \frac{du}{\underset{1}{\cancel{e^x}}} = \int \frac{1}{u} \, du$

$= \ln|e^x+1| + C$.

⑲ $\int x^7 \cos(x^8+2)\,dx \; ; \; u = x^8+2$

$= \int \cancel{x^7}^{\,1} \cos u \, \frac{du}{8\cancel{x^7}_{\,1}} = \frac{1}{8}\int \cos u \, du$

$= \frac{1}{8} \sin u + C$

$= \frac{1}{8} \sin(x^8+2) + C$

⑳ $\int \frac{\tan^5 x}{\cos^2 x}\,dx \; ;$

$$\begin{tikzcd}
\sin^2 A + \cos^2 A = 1 \arrow[d, "\div\cos^2 A"'] \arrow[dd, bend left, "\div\sin^2 A"] \\
\tan^2 A + 1 = \sec^2 A \\
1 + \cot^2 A = \operatorname{cosec}^2 A
\end{tikzcd}$$

$= \int \sec^2 x \, (\tan x)^5 \, dx \; ; \; u = \tan x$

$= \int \sec^2 x \; u^5 \, \frac{du}{\sec^2 x} = \frac{u^6}{6} + C$

$= \frac{\tan^6 x}{6} + C$

㉑ $\int \frac{1}{\sin^2 x \, \tan x}\,dx$

$= \int \operatorname{cosec}^2 x \, \cot x \, dx \; ; \; u = \cot x$

$= \int \cancel{\operatorname{cosec}^2 x}^{\,1} \; u \, \frac{du}{-\cancel{\operatorname{cosec}^2 x}_{\,1}} = -\int u \, du = -\frac{u^2}{2} + C$

$= -\frac{\cot^2 x}{2} + C$

㉒ $$\int \frac{\sin x \cos x}{\sin^2 x + 7\cos^2 x}\, dx \quad ; \quad u = (\sin x)^2 + 7(\cos x)^2$$

$$\frac{du}{dx} = 2\sin x \cos x + 14 \cos x(-\sin x)$$

$$= -12 \sin x \cos x$$

$$= \int \frac{\cancel{\sin x \cos x}}{u} \frac{du}{-12\,\cancel{\sin x}\cos x}$$

$$= -\frac{1}{12}\int \frac{1}{u}\, du$$

$$= -\frac{1}{12}\ln\left|\sin^2 x + 7\cos^2 x\right| + C$$

㉓ $$\int \frac{\sin 2x}{\sqrt{5\sin^2 x + 4\cos^2 x}}\, dx \quad ; \quad u = 5(\sin x)^2 + 4(\cos x)^2$$

$$\frac{du}{dx} = 10 \sin x \cos x + 8\cos x(-\sin x)$$

$$= \int \frac{2\sin x \cos x}{\sqrt{u}} \frac{du}{2\sin x \cos x}$$

$$= \int u^{-\frac{1}{2}}\, du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{5\sin^2 x + 4\cos^2 x} + C$$

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㉔ $\int \overset{E}{e^{-x}} \overset{T}{\sin x}\, dx$; $u = \sin x$, $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = \cos x \leftrightarrow v = -e^{-x}$

$$- \int v \frac{du}{dx} dx$$

$$= \underbrace{(\sin x)}_{u}\underbrace{(-e^{-x})}_{v} - \int -e^{-x} \cos x \, dx$$

$u_1 = \cos x$, $\frac{dv}{dx} = e^{-x}$

$\frac{du_1}{dx} = -\sin x$, $v = -e^{-x}$

$$= -e^{-x} \sin x + \int e^{-x} \cos x \, dx$$

$$v \quad u \quad - \int v \frac{du_1}{dx} dx$$

$$= -e^{-x} \sin x + \left[-e^{-x} \cos x - \int e^{-x} \sin x \, dx \right]$$

$$\underbrace{\int e^{-x} \sin x \, dx}_{I} = -e^{-x} \sin x - e^{-x} \cos x - \underbrace{\int e^{-x} \sin x \, dx}_{I}$$

$$2I = -e^{-x}(\sin x + \cos x)$$

$$I = -\frac{1}{2} e^{-x} (\sin x + \cos x)$$

$$\int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C$$

㉕ $$\int \sin x \, e^{\cos x} \, dx \quad ; \; u = \cos x$$

$$= \int \overset{1}{\cancel{\sin x}} \, e^{u} \, \frac{du}{\cancel{-\sin x} \; -1}$$

$$= -e^{u} + C$$

$$= -e^{\cos x} + C$$

Method I ($e^{\ln a} = a$)

㉖ $$\int 2^{x} \, dx \quad ; \quad e^{\ln a} = a.$$

$$= \int e^{\ln 2^{x}} \, dx$$

$$= \int e^{x \overset{a}{\underline{\ln 2}}} \, dx \qquad \int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$= \frac{e^{x \ln 2}}{\ln 2} + C$$

$$= \frac{2^{x}}{\ln 2} + C$$

Method II ($u = 2^{x}$)

$$\int 2^{x} \, dx \quad ; \; u = 2^{x}$$

$$= \int u \, \frac{du}{2^{x} \ln 2}$$

$$= \frac{1}{\ln 2} \int \frac{u}{2^{x}} \, du = \frac{1}{\ln 2} \int \frac{u}{u} \, du$$

$$= \frac{1}{\ln 2} \int 1 \, du$$

$$= \frac{1}{\ln 2} \left(2^{x} \right) + C$$

㉗ $$\int \frac{1}{4-9x}\,dx = \frac{\ln|4-9x|}{-9} + C$$

$$= -\frac{1}{9}\ln|4-9x| + C$$

㉘ $$\int \frac{1}{4-9x^2}\,dx = \int \frac{1}{2^2-\underset{\longleftarrow}{(3x)^2}}\,dx$$

$$= \frac{\frac{1}{2(2)}\ln\left(\frac{2+3x}{2-3x}\right)}{3} + C$$

$$= \frac{1}{12}\ln\left(\frac{2+3x}{2-3x}\right) + C$$

㉙ $$\int \frac{x}{4-9x^2}\,dx \quad ; \quad u = 4-9x^2$$

$$= \int \frac{\overset{1}{\cancel{x}}}{u}\,\frac{du}{-18\underset{1}{\cancel{x}}} = -\frac{1}{18}\int \frac{1}{u}\,du$$

$$= -\frac{1}{18}\ln|4-9x^2| + C$$

㉚ $$\int \frac{x+3}{4-9x^2}\,dx$$

$$= \int \frac{x}{4-9x^2} + \frac{3}{4-9x^2}\,dx$$

$$= \underbrace{\int \frac{x}{4-9x^2}\,dx}_{\downarrow \text{ from } ㉙} + \underbrace{3\int \frac{1}{2^2-(3x)^2}\,dx}_{\downarrow \text{ from } ㉘}$$

$$= -\frac{1}{18}\ln|4-9x^2| + 3\left(\frac{1}{12}\ln\left(\frac{2+3x}{2-3x}\right)\right) + C$$

$$= -\frac{1}{18}\ln|4-9x^2| + \frac{1}{4}\ln\left(\frac{2+3x}{2-3x}\right) + C$$

㉛ $$\int \frac{x^2}{4-9x^2}\,dx$$

$$\begin{array}{r|l} & -\frac{1}{9} \\ \hline -9x^2+4 & x^2 \\ & -\left(x^2 - \frac{4}{9}\right) \\ \hline & \frac{4}{9} \end{array}$$

$$= \int -\frac{1}{9} + \frac{4}{9}\left(\frac{1}{4-9x^2}\right)dx$$

$$= -\frac{1}{9}\int 1\,dx + \underbrace{\frac{4}{9}\int \frac{1}{4-9x^2}\,dx}_{\downarrow \text{ Qn } 28}$$

$$= -\frac{1}{9}x + \frac{4}{9}\left(\frac{1}{12}\ln\left(\frac{2+3x}{2-3x}\right)\right) + C$$

$$= -\frac{1}{9}x + \frac{1}{12}\ln\left(\frac{2+3x}{2-3x}\right) + C$$

㉜ $$\int \frac{x}{\sqrt{4-9x^2}}\,dx \; ; \text{ let } u = 4-9x^2$$

$$= \int \frac{\overset{1}{\cancel{x}}}{\sqrt{u}} \, \frac{du}{-18\underset{1}{\cancel{x}}} = -\frac{1}{18}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{18}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$= -\frac{1}{9}\sqrt{4-9x^2} + C$$

㉝ $$\int \frac{1}{\sqrt{1-9x}}\,dx$$

$$= \int (1-9x)^{-\frac{1}{2}}\,dx$$

$$= \frac{(1-9x)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)(-9)} + C$$

$$= -\frac{2}{9}\sqrt{1-9x} + C$$

③④ $$\int \frac{1}{\sqrt{1-9x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1^2-(\underline{\underline{3}}x)^2}}\,dx$$

$$= \frac{\sin^{-1}(3x)}{3} + C$$

$$= \frac{1}{3}\sin^{-1}(3x) + C$$

$$\int \sin 7x\,dx = -\frac{\cos 7x}{7} + C$$

$\downarrow$ $u = 7x$

$$\int \sin u \,\frac{du}{7} = \frac{1}{7}\int \sin u\,du$$

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㉟ $$\int \frac{x}{\sqrt{1-9x}}\,dx$$

$$= \int \overset{A}{x}\,\overset{A}{(1-9x)}^{-\frac{1}{2}}\,dx$$

$u = x \qquad \frac{dv}{dx} = (1-9x)^{-\frac{1}{2}}$

$\frac{du}{dx} = 1 \qquad v = \frac{(1-9x)^{\frac{1}{2}}}{(\frac{1}{2})(-9)}$

$= -\frac{2}{9}\sqrt{1-9x}$ ✓

$$= x\cdot\left(-\frac{2}{9}\sqrt{1-9x}\right) - \int -\frac{2}{9}\sqrt{1-9x}\cdot 1\,dx$$

$$= -\frac{2}{9}x\sqrt{1-9x} + \frac{2}{9}\int (1-9x)^{\frac{1}{2}}\,dx$$

$$= -\frac{2}{9}x\sqrt{1-9x} + \frac{2}{9}\,\frac{(1-9x)^{\frac{3}{2}}}{(\frac{3}{2})(-9)} + C$$

$$= -\frac{2}{9}x\sqrt{1-9x} - \frac{4}{243}\sqrt{1-9x}^{\,3} + C$$

(36) $$\int \frac{x+2}{\sqrt{1-8x-4x^2}}\,dx$$; $u = 1-8x-4x^2$

$\frac{du}{dx} = -8-8x$

$$= \int \frac{A(-8-8x)+B}{\sqrt{1-8x-4x^2}}\,dx$$

$x+2 = A(-8-8x)+B$

$= -8Ax - 8A + B$

$-8A = 1$, $-8A+B = 2$

$A = -\frac{1}{8}$, $B = 2+8A$

$= 2+8(-\frac{1}{8})$

$B = 1$

$$= -\frac{1}{8}\int \frac{-8-8x}{\sqrt{1-8x-4x^2}} + \frac{1}{\sqrt{1-8x-4x^2}}\,dx$$

$$= -\frac{1}{8}\int \frac{\cancel{-8-8x}^{\,1}}{\sqrt{u}}\,\frac{du}{\cancel{-8-8x}_{\,1}} + \int \frac{1}{\sqrt{\sqrt{5}^2-[2(x+1)]^2}}\,dx$$

$1-8x-4x^2$

$= 1-4(x^2+2x)$

$= 1-4[(x+1)^2-1^2]$

$= 5-4(x+1)^2$

$\int u^{-\frac{1}{2}}du$

$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}}$

$= 2\sqrt{u}$

$$= -\frac{1}{8}\left(2\sqrt{1-8x-4x^2}\right) + \frac{1}{2}\sin^{-1}\left(\frac{2(x+1)}{\sqrt{5}}\right) + C$$

$$= -\frac{1}{4}\sqrt{1-8x-4x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2(x+1)}{\sqrt{5}}\right) + C$$

㊲ $\int \frac{x}{x^2+4x+7}\,dx = \int \frac{A(2x+4)+B}{x^2+4x+7}\,dx$; $x = A(2x+4)+B$

$x = 2Ax + 4A + B$

$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+7}\,dx - 2\int \frac{1}{(x+2)^2-2^2+7}\,dx$

$2A = 1$, $4A+B=0$

$A = \frac{1}{2}$, $2+B=0$

$B = -2$

$= \frac{1}{2}\ln\left|x^2+4x+7\right| - 2\int \frac{1}{(x+2)^2+\sqrt{3}^2}\,dx$

$= \frac{1}{2}\ln\left|x^2+4x+7\right| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + C$.

㊳ $\int \frac{\sqrt{x}}{1+x}\,dx$; $u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du$

$= \int \frac{\sqrt{x}}{1+x}\,2\sqrt{x}\,du$

$= 2\int \frac{x}{1+x}\,du$; $x = u^2$

$= 2\int \frac{u^2+1-1}{1+u^2}\,du$

$\downarrow$ Long Divn.

$= 2\int 1 - \frac{1}{1+u^2}\,du$

$= 2[u - \tan^{-1}u) + c$

$= 2[\sqrt{x} - \tan^{-1}\sqrt{x}] + C$

(39) $$\int \frac{x^3}{\sqrt{x^2+4}}\,dx$$

$$= \int \overset{u}{x^2} \cdot \overbrace{\frac{x}{\sqrt{x^2+4}}}^{\frac{dv}{dx}}\,dx$$

$u = x^2$, $\frac{dv}{dx} = \frac{x}{\sqrt{x^2+4}}$, $v = \int \frac{x}{\sqrt{x^2+4}}\,dx$

$\frac{du}{dx} = 2x$

$$v = \int \frac{x}{\sqrt{k}}\,\frac{dk}{2x}$$

$$= \frac{1}{2}\int k^{-\frac{1}{2}}\,dk = \frac{1}{2}\left(\frac{k^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$v = \sqrt{x^2+4}$$

$$= x^2\sqrt{x^2+4} - \int 2x\sqrt{x^2+4}\,dx$$

l- $p = x^2+4$

$$= x^2\sqrt{x^2+4} - \int \cancel{2x}\sqrt{p}\,\frac{dp}{\cancel{2x}}$$

$$\int \sqrt{p}\,dp = \frac{p^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= x^2\sqrt{x^2+4} - \frac{2}{3}\sqrt{x^2+4}^{\,3} + C$$

(40) $$\int e^{2x}\sqrt{e^{2x}+1}\,dx \; ; \; u = e^{2x}+1$$

$$= \int \cancel{e^{2x}}\sqrt{u}\,\frac{du}{2\cancel{e^{2x}}} = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{3}\sqrt{e^{2x}+1}^{\,3} + C$$

$x-3 = A(2x-2)+B$

$x-3 = 2Ax-2A+B$

④① $$\int \frac{x-3}{x^2-2x+4}\,dx = \int \frac{A(2x-2)+B}{x^2-2x+4}\,dx$$

$2A=1$, $-3=-2A+B$

$A=\frac{1}{2}$, $B=-3+2A$

$=-2$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}\,dx - 2\int \frac{1}{(x-1)^2+\sqrt{3}^3}\,dx$$

$$= \frac{1}{2}\ln|x^2-2x+4| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C$$ ✓.

④② $$\int \sqrt{\frac{1-3x}{1+3x}}\,dx$$

$$= \int \sqrt{\frac{(1-3x)(1-3x)}{(1+3x)(1-3x)}}\,dx$$

$$= \int \sqrt{\frac{(1-3x)^2}{1^2-(3x)^2}}\,dx$$

$$= \int \frac{\sqrt{(1-3x)^2}}{\sqrt{1-(3x)^2}}\,dx = \int \frac{1-3x}{\sqrt{1-(3x)^2}}\,dx$$

$$= \underbrace{\int \frac{1}{\sqrt{1^2-(3x)^2}}\,dx}_{\text{MF 27}} - \underbrace{3\int \frac{x}{\sqrt{1-9x^2}}\,dx}_{\text{cancully.}}$$

$$= \frac{\sin^{-1}\left(\frac{3x}{1}\right)}{3} \cancel{-} \overset{1}{\cancel{3}}\int \frac{\overset{1}{\cancel{x}}}{\sqrt{u}}\,\frac{du}{\underset{6}{\cancel{-18}}\cancel{x}\,1}$$

$$\int u^{-\frac{1}{2}}du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{u}$$

$$= \frac{1}{3}\sin^{-1}(3x) + \frac{1}{3}\sqrt{1-9x^2} + C$$ ✓.

x+x. ㊸

$$\int \tan^2 x \sec^4 x \, dx$$

$$\cos^2 A + \sin^2 A = 1$$

$\downarrow \div \cos^2 A$

$$\overset{\checkmark}{1} + \overset{\otimes}{\overbrace{\tan^2 A}} = \underset{\checkmark}{\sec^2 A}$$

$$= \int \tan^2 x \quad \sec^2 x \quad \sec^2 x \quad dx$$

$$= \int \tan^2 x \left(1 + \tan^2 x\right) \sec^2 x \, dx \quad ; \text{ let } \quad u = \tan x$$

$$= \int u^2(1+u^2) \, \cancel{\sec^2 x}^{\,1} \, \frac{du}{\cancel{\sec^2 x}_{\,1}}$$

$$= \int u^2 + u^4 \, du = \frac{u^3}{3} + \frac{u^5}{5} + c$$

$$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$$

㊹

$$\int \frac{\left(e^{10x} - 1\right)e^{-5x}}{\left(e^{10x} + 1\right)e^{-5x}} \, dx$$

$$= \int \frac{e^{10x} \cdot e^{-5x} - e^{-5x}}{e^{10x} \cdot e^{-5x} + e^{-5x}} \, dx$$

$$= \int \frac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}} \, dx \quad ; \quad u = e^{5x} + e^{-5x}$$

$$= \int \frac{\cancel{e^{5x} - e^{-5x}}}{u} \; \frac{\overset{1}{du}}{\cancel{5e^{5x} - 5e^{-5x}}_{\,5}} = \frac{1}{5}\int \frac{1}{u} \, du$$

$$= \frac{1}{5}\ln\left|e^{5x} + e^{-5x}\right| + c$$

$\sin 2A = 2\cos A \sin A$

④g) $$\int \frac{e^{\sin^2 2x} \sin 4x}{\sqrt{1+e^{\sin^2 2x}}}\, dx$$

$u = \boxed{1+e^{\boxed{(\sin 2x)}^2}}$

$\frac{du}{dx} = e^{\sin^2 2x} \cdot (2\sin 2x)(\cos 2x)(2)$

$= 2e^{\sin^2 2x} \sin 4x$

$$= \int \frac{\cancel{e^{\sin^2 2x} \sin 4x}^{1}}{\sqrt{u}} \frac{du}{2\cancel{e^{\sin^2 2x} \sin 4x}_{1}}$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}} du = \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c$$

$$= \sqrt{1+e^{\sin^2 2x}} + c$$

④f) $$\int \overset{A}{x} \, \overset{I}{\tan^{-1}(3x)}\, dx$$

$u = \tan^{-1}(3x)$ $\quad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{1+(3x)^2} \cdot 3$, $\quad v = \frac{x^2}{2}$

$= \frac{3}{1+9x^2}$

$$= \frac{x^2}{2}\tan^{-1}(3x) - \int \frac{x^2}{2} \cdot \frac{3}{1+9x^2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1}(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\, dx \quad ;$$

$$\begin{array}{r|l} & \frac{1}{9} \\ 9x^2+1 & x^2 \\ & -(x^2 + \frac{1}{9}) \\ \hline & -\frac{1}{9} \end{array}$$

$$= \frac{x^2}{2}\tan^{-1}(3x) - \frac{3}{2}\int \frac{1}{9} - \frac{1}{9} \cdot \frac{1}{1+(3x)^2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1}(3x) - \frac{3}{2}\left(\frac{1}{9}\right)\int 1 - \frac{1}{1+(3x)^2}\, dx$$

$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}\left[x - \frac{1}{3}\tan^{-1}(3x)\right] + c$$

㊼ Method1 (LIA$\overset{u}{\text{TE}}$)

$$\int x\cos(3x^2)e^{x^2}\,dx = \int \overset{u}{\cos(3x^2)}\cdot \overset{\frac{dv}{dx}}{xe^{x^2}}\,dx$$

$u=\cos(3x^2)$, $\frac{dv}{dx}=xe^{x^2}$

$\frac{du}{dx}=-6x\sin(3x^2)$, $v=\frac{1}{2}\int 2x\,e^{x^2}\,dx$

$v=\frac{1}{2}e^{x^2}$

$$= \cos(3x^2)\cdot\frac{1}{2}e^{x^2} - \int -6x\sin(3x^2)\cdot\frac{1}{2}e^{x^2}\,dx$$

$$= \frac{1}{2}e^{x^2}\cos(3x^2) + 3\left[\int \overset{\frac{dv}{dx}}{xe^{x^2}}\,\overset{u}{\sin 3x^2}\,dx\right]$$

$u=\sin 3x^2$, $\frac{dv}{dx}=xe^{x^2}$

$\frac{du}{dx}=6x\cos 3x^2$, $v=\frac{1}{2}e^{x^2}$

$$= \frac{1}{2}e^{x^2}\cos(3x^2) + 3\left[\frac{1}{2}e^{x^2}\sin 3x^2 - \int 6x\cos 3x^2\cdot\frac{1}{2}e^{x^2}\,dx\right]$$

$$= \frac{1}{2}e^{x^2}\cos(3x^2) + 3\left[\frac{1}{2}e^{x^2}\sin 3x^2 - 3\int xe^{x^2}\cos 3x^2\,dx\right]$$

$$\int x\cos(3x^2)e^{x^2}\,dx = \frac{1}{2}e^{x^2}\cos 3x^2 + \frac{3}{2}e^{x^2}\sin 3x^2 - 9\int xe^{x^2}\cos 3x^2\,dx$$

$$10\int x\cos(3x^2)e^{x^2}\,dx = \frac{1}{2}e^{x^2}\left[\cos 3x^2 + 3\sin 3x^2\right]$$

$$\int x\cos(3x^2)\,e^{x^2}\,dx = \frac{1}{20}e^{x^2}\left(\cos 3x^2 + 3\sin 3x^2\right) + C$$

~~C.~~

Method II (using $u = 3x^2$)

⑷⑺ $$\int x \cos(3x^2)\, e^{x^2}\, dx \quad ; \; u = 3x^2$$

$$= \int \cancel{x}^{1} \cos u \; e^{\frac{u}{3}} \frac{du}{6\cancel{x}_{1}}$$

$$= \frac{1}{6}\int (\cos u)\, e^{\frac{u}{3}}\, du.$$

$u_1 = \cos u \qquad \frac{dv_1}{du} = e^{u}$

$\frac{du_1}{du} = -\sin u \qquad v_1 = e^{u}$

from ①

$$= \frac{1}{6}\left[\frac{3e^{\frac{u}{3}}}{10}(\cos u + 3\sin u)\right] + C$$

$$= \frac{1}{20} e^{x^2}\left[\cos(3x^2) + 3\sin(3x^2)\right] + C$$

$u = \cos x, \; \frac{dv}{dx} = e^{\frac{x}{3}}$

$$\int e^{\frac{x}{3}} \cos x \, dx \qquad \frac{du}{dx} = -\sin x, \; v = 3e^{\frac{x}{3}}$$

$$= 3e^{\frac{x}{3}}\cos x + 3\int e^{\frac{x}{3}} \sin x \, dx \qquad u_1 = \sin x, \; \frac{dv}{dx} = e^{\frac{x}{3}}$$

$\frac{du_1}{dx} = \cos x, \; v = 3e^{\frac{x}{3}}$

$$\underbrace{\int e^{\frac{x}{3}} \cos x \, dx}_{I} = 3e^{\frac{x}{3}}\cos x + 3\left[3e^{\frac{x}{3}}\sin x - 3\underbrace{\int e^{\frac{x}{3}}\cos x \, dx}_{I}\right]$$

$$\int e^{\frac{x}{3}} \cos x \, dx = 3e^{\frac{x}{3}}\cos x + 9e^{\frac{x}{3}}\sin x - 9\int e^{\frac{x}{3}}\cos x \, dx$$

$$10\int e^{\frac{x}{3}} \cos x \, dx = 3e^{\frac{x}{3}}(\cos x + 3\sin x)$$

$$\int e^{\frac{x}{3}} \cos x \, dx = \frac{3}{10} e^{\frac{x}{3}}(\cos x + 3\sin x) \quad — ①$$

⑱ $\int \frac{\sin x}{1+\cos x}\,dx \quad ; \qquad u = 1+\cos x$

$= \int \frac{\sin x}{u} \frac{du}{-\sin x} = -\int \frac{1}{u}\,du$

$= -\ln\left|1+\cos x\right| + C$

⑲ $\int \sin^{-1}(3x)\,dx = \int \underset{\overset{\uparrow}{\frac{dv}{dx}}}{\underline{1}} \cdot \underset{\overset{\uparrow}{u}}{\underline{\sin^{-1}(3x)}}\,dx$

$u = \sin^{-1}(3x) \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 \qquad v = x$

$= x\sin^{-1}(3x) - \int \frac{3x}{\sqrt{1-9x^2}}\,dx$

$= x\sin^{-1}(3x) - \int \frac{\overset{1}{\cancel{3x}}}{\sqrt{u}} \frac{du}{\underset{-6}{\cancel{-18x}}}$

$= x\sin^{-1}(3x) + \frac{1}{6}\int u^{-\frac{1}{2}}\,du$

$= x\sin^{-1}(3x) + \frac{1}{6}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$

$= x\sin^{-1}(3x) + \frac{1}{3}\sqrt{1-9x^2} + C$

㊿ $$\int x \sin(3x^2+1)\, dx \;; \qquad u = 3x^2+1$$

$$= \int \cancel{x} \sin u \, \frac{du}{6\cancel{x}} = \frac{1}{6}\int \sin u \, du$$

$$= -\frac{1}{6}\cos(3x^2+1) + C \,.$$

(51) $$\int x \sin^{-1}\left(\frac{2}{x}\right) dx \;; \qquad u = \boxed{\sin^{-1}\left(\boxed{\frac{2}{x}}\right)} \,, \quad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-\left(\frac{2}{x}\right)^2}} \cdot \left(-\frac{2}{x^2}\right) \,, \quad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2}\sin^{-1}\left(\frac{2}{x}\right) + \int \frac{\cancel{x^2}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{x^2}\sqrt{1-\frac{4}{x^2}}}\, dx$$

$$= \frac{1}{2}x^2\sin^{-1}\left(\frac{2}{x}\right) + \int \frac{1}{\sqrt{\frac{x^2-4}{x^2}}}\, dx$$

$$= \frac{1}{2}x^2\sin^{-1}\left(\frac{2}{x}\right) + \int \frac{x}{\sqrt{x^2-4}}\, dx$$

$$k = x^2-4$$

$$\int \frac{x}{\sqrt{k}} \frac{dk}{2x}$$

$$= \frac{1}{2}\int k^{-\frac{1}{2}}\, dk$$

$$= \frac{1}{2}\left(\frac{k^{\frac{1}{2}}}{\frac{1}{2}}\right) = k^{\frac{1}{2}}$$

$$= \frac{1}{2}x^2\sin^{-1}\left(\frac{2}{x}\right) + \sqrt{x^2-4} + C \,.$$

(52) $$\int \frac{1}{x^4-1}\,dx$$

$$= \int \frac{1}{(x^2)^2-1^2}\,dx$$

$$= \int \frac{1}{(x^2-1)(x^2+1)}\,dx$$

$$= \int \frac{1}{(x-1)(x+1)(x^2+1)}\,dx$$

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

$x=1$: $A = \frac{1}{2(1^2+1)} = \frac{1}{4}$, $x=-1$: $B = \frac{1}{-2(2)} = -\frac{1}{4}$

$x=0$: $1 = \frac{1}{4} - \frac{1}{4}(-1) + D(-1)(1) \Rightarrow -D = \frac{1}{2}$

$D = -\frac{1}{2}$

coeffnt of x^3: $0 = A + B + C$

$C = -A - B = -\frac{1}{4} - \left(-\frac{1}{4}\right) = 0$

$$= \int \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}\,dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1}x + C$$

(53) $$\int \frac{1}{\sqrt{x}(1+\sqrt{x}^2)}\,dx$$

$$= \int \frac{1}{\sqrt{x}} \cdot \frac{1}{(1+\sqrt{x}^2)}\,dx \; ; \quad u = \sqrt{x}$$

$$= \int \cancel{\frac{1}{\sqrt{x}}}\,1 \cdot \frac{1}{1+u^2}\,\frac{du}{\frac{1}{2}\cdot\cancel{\frac{1}{\sqrt{x}}}\,1}$$

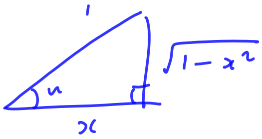
$$= 2\int \frac{1}{1+u^2}\,du = 2\tan^{-1}u + c$$

$$= 2\tan^{-1}\sqrt{x} + C$$

(54) $\int \frac{x^3}{\sqrt{1-x^2}} dx$

$x = \cos u \implies$

$\frac{dx}{du} = -\sin u$



$\sin u = \sqrt{1-x^2}$

$= \int \frac{x^3}{\sqrt{1-x^2}} (-\sin u)\, du$

$= -\int \frac{\cos^3 u}{\sqrt{1-\cos^2 u}} \sin u \quad du$; $\quad \cos^2 A + \sin^2 A = 1$

$= -\int \frac{\cos^3 u}{\cancel{\sqrt{\sin^2 u}}_{1}} \cancel{\sin u}^{1} \quad du$

$= -\int \cos^3 u \quad du$

let $k = \sin u$

$\frac{dk}{du} = \cos u$

$= -\int \cos^{\cancel{3}\,2} u \quad \frac{dk}{\cancel{\cos u}\ 1}$

$= -\int \cos^2 u \quad dk$

$= -\int 1 - \sin^2 u \quad dk$

$= -\int 1 - k^2 \quad dk$

$= -\left[k - \frac{k^3}{3} \right] + C$

$= -\left[\sin u - \frac{\sin^3 u}{3} \right] + C$

$= -\sqrt{1-x^2} + \frac{1}{3} \sqrt{1-x^2}^{\,3} + C$ ✓

(55) $\overbrace{\int e^{3x}\cos 2x\ dx}^{I}$

$u = \cos 2x$ $\qquad \frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = -2\sin 2x$ $\qquad v = \frac{1}{3}e^{3x}$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x}\sin 2x\ dx$

$u_1 = \sin 2x$ $\qquad \frac{dv}{dx} = e^{3x}$

$\frac{du_1}{dx} = 2\cos 2x$ $\qquad v = \frac{1}{3}e^{3x}$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\left[\frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\underbrace{\int e^{3x}\cos 2x\ dx}_{I}\right]$

$I = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x - \frac{4}{9}I$

$\frac{13}{9}I = \frac{1}{9}e^{3x}\left(3\cos 2x + 2\sin 2x\right)$

$\therefore \int e^{3x}\cos 2x\ dx = \frac{1}{13}e^{3x}\left(3\cos 2x + 2\sin 2x\right) + C$ #

(56) $$\int \frac{1}{(1+x^2)^2}\, dx \; ; \qquad x = \tan u$$

$$\frac{dx}{du} = \sec^2 u . \Rightarrow dx = \sec^2 u \, du$$

$$= \int \frac{1}{(1+\tan^2 u)^2} \sec^2 u \quad du$$

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$ $\big) \div \cos^2 A$

$$= \int \frac{1}{(\sec^2 u)^{\cancel{2}}} \cancel{\sec^2 u} \; du$$

$\cos 2A = 2\cos^2 A - 1$

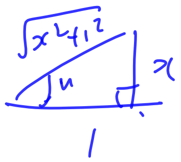
$\cos^2 A = \frac{1}{2}(\cos 2A + 1)$

$$= \int \frac{1}{\sec^2 u} \, du = \int \cos^2 u \, du$$

$$= \frac{1}{2} \int \cos 2u + 1 \quad du$$

$$= \frac{1}{2}\left[\frac{\sin 2u}{2} + u \right] + C$$

$\tan u = \frac{x}{1}$



$$= \frac{1}{2}\left[\frac{2\sin u \cos u}{2} + u \right] + C$$

$\cos u = \frac{1}{\sqrt{1+x^2}}$

$\sin u = \frac{x}{\sqrt{1+x^2}}$

$$= \frac{1}{2}\left[\frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} + \tan^{-1} x \right] + C$$

$$= \frac{1}{2}\left[\frac{x}{1+x^2} + \tan^{-1} x \right] + C$$

(57) $$\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$$

$$= \int \frac{1}{1+x^2}\,e^{\tan^{-1}x}\,dx \quad ; \quad u = \tan^{-1}x$$

$$= \int \cancel{\frac{1}{1+x^2}}\,e^{u}\,\frac{du}{\cancel{\left(\frac{1}{1+x^2}\right)}} = e^{u}+c$$

$$= e^{\tan^{-1}x} + c$$

(58) $$\int \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}\,dx \quad ; \quad u=\sqrt{x}$$

$$= \int \frac{1}{\sqrt{x}}\cdot\frac{1}{e^{u}}\,\frac{du}{\frac{1}{2}\cdot\frac{1}{\sqrt{x}}}$$

$$= 2\int e^{-u}\,du = \frac{2e^{-u}}{-1} + c$$

$$= -2e^{-\sqrt{x}} + c$$

(59) $$\int \frac{1}{x(1-\ln x)^5}\,dx \quad ; u = 1-\ln x$$

$$= \int \cancel{\frac{1}{x}}\cdot\frac{1}{u^5}\,\frac{du}{-\cancel{\frac{1}{x}}} = -\int u^{-5}\,du = -\frac{u^{-4}}{(-4)} + c$$

$$= \frac{1}{4(1-\ln x)^4} + c$$

(60)

$$\int \frac{1}{\sqrt{x}-x}\,dx$$

$$= \int \frac{1}{\sqrt{x}(1-\sqrt{x})}\,dx$$

$$= \int \overset{1}{\cancel{\frac{1}{\sqrt{x}}}} \cdot \frac{1}{u}\,\frac{du}{-\frac{1}{2}\cdot\cancel{\frac{1}{\sqrt{x}}}\;1}$$

$$= -2\ln\left|1-\sqrt{x}\right| + C$$

(61)

$$\int \frac{1}{\sqrt{x}\sqrt{1-\sqrt{x}}}\,dx \quad ; \text{ let } u = 1-\sqrt{x}$$

$$= \int \overset{1}{\cancel{\frac{1}{\sqrt{x}}}} \cdot \frac{1}{\sqrt{u}}\,\frac{du}{-\frac{1}{2}\cdot\cancel{\frac{1}{\sqrt{x}}}}\;1 = -2\int u^{-\frac{1}{2}}\,du = \frac{-2u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= -4\sqrt{1-\sqrt{x}} + C$$

(62)

$$\int \frac{1}{\cos^2 x\sqrt{1+\tan x}}\,dx \quad ; \quad u = 1+\tan x$$

$$= \int \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$$

$$= \int \frac{\cancel{\sec^2 x}\;1}{\sqrt{u}}\,\frac{du}{\cancel{\sec^2 x}\;1} = \int u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{1+\tan x} + C$$

(63) $\int \tan^7 2x + \tan^5 2x \, dx$

$\sin^2 A + \cos^2 A = 1$ $\div \cos^2 A$

$\tan^2 A + 1 = \sec^2 A$

$= \int \tan^5 2x \left(\tan^2 2x + 1\right) dx$

$= \int (\tan 2x)^5 \sec^2 2x \, dx$; $u = \tan 2x$

$= \int u^5 \, \cancel{\sec^2 2x}^{\,1} \frac{du}{2\cancel{\sec^2 2x}_{\,1}}$

$= \frac{1}{2} \int u^5 du = \frac{1}{2}\left(\frac{u^6}{6}\right) + C$

$= \frac{1}{12} \tan^6 2x + C$

(64) $\int \cos^4 x \, dx$

$\cos 2A = 2\cos^2 A - 1$

$\cos^2 A = \frac{\cos 2A + 1}{2}$

$= \int (\cos^2 x)^2 dx$

$= \int \left[\frac{1}{2}(\cos 2x + 1)\right]^2 dx$

$= \frac{1}{4} \int \cos^2 2x + 2\cos 2x + 1 \quad dx$

$= \frac{1}{4} \int \frac{1}{2}(\cos 4x + 1) + 2\cos 2x + 1 \, dx$

$= \frac{1}{4} \int \frac{1}{2} \cos 4x + 2\cos 2x + \frac{3}{2} \, dx$

$= \frac{1}{4}\left[\frac{1}{8} \sin 4x + \sin 2x + \frac{3}{2} x\right] + C$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos^2 A = \frac{\cos 2A + 1}{2}$$

⑥⑤ $\int \sin^4 x \, dx$

$$= \int (\sin^2 x)^2 dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \, dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1) \, dx$$

$$= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \, dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + C$$

⑥⑥ $\int \sec^3 x \, dx$;

$$1 + \tan^2 A = \sec^2 A$$

$$= \int \sec x (\sec^2 x) \, dx$$

$$= \int \sec x (1 + \tan^2 x) \, dx$$

$$= \int \sec x + \sec x (\tan x)^2 \, dx$$

$$= \underbrace{\int \sec x \, dx}_{\downarrow \text{MF27}} + \int \overbrace{\tan x}^{u} \overbrace{(\sec x \tan x)}^{\frac{dv}{dx}} \, dx$$

$u = \tan x$ $\quad$ $\frac{dv}{dx} = \sec x \tan x$

$\frac{du}{dx} = \sec^2 x$ $\quad$ $v = \sec x$

$$= \ln(\sec x + \tan x) + \left[\sec x \tan x - \int \sec^2 x \cdot \sec x \, dx\right]$$

$$\int \sec^3 x\, dx = \ln(\sec x + \tan x) + \sec x \tan x - \int \sec^3 x\, dx$$

$$2\int \sec^3 x\, dx = \ln|\sec x + \tan x| + \sec x \tan x$$

$$\int \sec^3 x\, dx = \frac{1}{2}\left[\ln|\sec x + \tan x| + \sec x \tan x\right] + c$$

(67) $\int \sec^4 x\, dx$; $1 + \tan^2 x = \sec^2 x$

$$= \int \sec^2 x\left(\sec^2 x\right) dx$$

$$= \int \sec^2 x\left(1 + \tan^2 x\right) dx \quad ; \quad u = \tan x$$

$$= \int \cancel{\sec^2 x}\left(1 + u^2\right) \frac{du}{\cancel{\sec^2 x}}$$

$$= u + \frac{u^3}{3} + C$$

$$= \tan x + \frac{1}{3}\tan^3 x + C$$

⑥⑧ $\int \tan^4(3x)\,dx$; $\overset{✓}{1} + \overset{\otimes}{\tan^2 A} = \overset{✓}{\sec^2 A}$

$$= \int \tan^2(3x)\cdot\tan^2(3x)\;dx$$

$$= \int (\sec^2 3x - 1)\;\tan^2 3x\;dx$$

$$= \int \underbrace{\sec^2 3x\,(\tan 3x)^2}_{\text{Cancely}} \;-\; \underbrace{\tan^2 3x}_{}\;dx$$

u=tan3x

$$= \int \cancel{\sec^2 3x}\,(u)^2\,\frac{du}{3\,\cancel{\sec^2 3x}} \;-\; \int \sec^2 3x - 1\;dx$$

$$= \frac{1}{3}\int u^2\,du \;-\; \left[\frac{\tan 3x}{3} - x\right] \;+ c$$

$$= \frac{1}{9}\tan^3 3x - \frac{1}{3}\tan 3x + x \quad + C$$

(69) $\int \sin^3(2x-1)\ dx$; $u = \cos(2x-1)$

$= \int \sin^3(2x-1)\ \frac{du}{-2\sin(2x-1)}$

$= -\frac{1}{2}\int \sin^2(2x-1)\ du$

$= -\frac{1}{2}\int 1 - \cos^2(2x-1)\ du$

$= -\frac{1}{2}\int 1 - u^2\ du$

$= -\frac{1}{2}\left[u - \frac{u^3}{3}\right] + c$

$= -\frac{1}{2}\left[\cos(2x-1) - \frac{1}{3}\cos^3(2x-1)\right] + c$.

(70) $\int \cos^3(1-x)\,dx$; $u = \sin(1-x)$

$$= \int \cos^{\overset{2}{\cancel{3}}}(1-x)\ \frac{du}{-\cancel{\cos(1-x)}\ 1}$$

$$= -\int \cos^2(1-x)\ du$$

$$= -\int 1-\sin^2(1-x)\ du$$

$$= -\int 1-u^2\ du$$

$$= -\left[u - \frac{u^3}{3}\right] + C$$

$$= -\sin(1-x) + \frac{1}{3}\sin^3(1-x) + C$$

⑺① $\int \cos^3(4x)\ \sin^8(4x)\ dx$; $u = \sin(4x)$

$$= \int \cos^{\cancel{3}\,2}(4x)\ u^8\ \frac{du}{4\cancel{\cos 4x}} = \frac{1}{4}\int \cos^2(4x)\ u^8\ du$$

$$= \frac{1}{4}\int \left(1 - \sin^2 4x\right) u^8\ du$$

$$= \frac{1}{4}\int (1-u^2)\ u^8\ du = \frac{1}{4}\int u^8 - u^{10}\ du$$

$$= \frac{1}{4}\left[\frac{u^9}{9} - \frac{u^{11}}{11}\right] + c$$

$$= \frac{1}{4}\left[\frac{\sin^9(4x)}{9} - \frac{\sin^{11}(4x)}{11}\right] + C$$

⑦②

$$\int \frac{1+x}{\sqrt{2-x-x^2}}\,dx = \int \frac{A(-1-2x)+B}{\sqrt{2-x-x^2}}\,dx$$

$A(-1-2x)+B = x+1$

$-2A = 1$, $-A+B = 1$

$A = -\frac{1}{2}$ $B = 1+A = \frac{1}{2}$

$$= -\frac{1}{2}\int \frac{-1-2x}{\sqrt{2-x-x^2}}\,dx + \frac{1}{2}\int \frac{1}{\sqrt{2-x-x^2}}\,dx$$

$u = 2-x-x^2$

$-x^2-x+2$

$= -[x^2+x]+2$

$= -[(x+\frac{1}{2})^2-(\frac{1}{2})^2]+2$

$= \frac{9}{4}-(x+\frac{1}{2})^2$

$$= -\frac{1}{2}\int \frac{\overset{1}{\cancel{-1-2x}}}{\sqrt{u}}\,\frac{du}{\underset{1}{\cancel{-1-2x}}} + \frac{1}{2}\int \frac{1}{\sqrt{(\frac{3}{2})^2-(x+\frac{1}{2})^2}}\,dx$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du + \frac{1}{2}\sin^{-1}\left(\frac{2(x+\frac{1}{2})}{2(\frac{3}{2})}\right) + C$$

$$= \sqrt{2-x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x+1}{3}\right) + C$$

(73) $$\int \frac{4x-1}{x^2+2x+3}\,dx = \int \frac{A(2x+2)+B}{x^2+2x+3}\,dx$$

$4x-1 = A(2x+2)+B$

$2A=4$, $2A+B=-1$

$A=2$ $\quad B=-1-2A$

$\quad\quad = -5$

$$= 2\int \frac{2x+2}{x^2+2x+3}\,dx - 5\int \frac{1}{(x+1)^2+\sqrt{2}^2}\,dx$$

$$= 2\ln|x^2+2x+3| - \frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

(74) $$\int \frac{5-3x}{x^2+8x+6}\,dx = \int \frac{A(2x+8)+B}{x^2+8x+6}\,dx$$

$5-3x = A(2x+8)+B$

$2A=-3$, $5 = 8A+B$

$A = -\frac{3}{2}$, $B = 5-8A$

$\quad\quad = 17$

$$= -\frac{3}{2}\int \frac{2x-8}{x^2+8x+6}\,dx + 17\int \frac{1}{(x+4)^2-\sqrt{10}^2}\,dx$$

$$= -\frac{3}{2}\ln|x^2+8x+6| + 17\cdot\frac{1}{2\sqrt{10}}\ln\left(\frac{x+4-\sqrt{10}}{x+4+\sqrt{10}}\right) + C$$

$$= -\frac{3}{2}\ln|x^2+8x+6| + \frac{17}{2\sqrt{10}}\ln\left(\frac{x+4-\sqrt{10}}{x+4+\sqrt{10}}\right) + C$$

(75) $\int \cos^5 x \; \sin^4 x \; dx$; let $u = \sin x$

$$= \int \cos^{\cancel{5}\,4} x \; u^4 \; \frac{du}{\cancel{\cos x}},$$

$$= \int \cos^4 x \; u^4 \; du$$

$$= \int (\cos^2 x)^2 \; u^4 \; du$$

$$= \int (1-\sin^2 x)^2 \; u^4 \; du$$

$$= \int (1-u^2)^2 \; u^4 \; du$$

$$= \int (1-2u^2+u^4)\, u^4 \, du$$

$$= \int u^4 - 2u^6 + u^8 \quad du$$

$$= \frac{u^5}{5} - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

(76) $\displaystyle\int \sqrt{\frac{1-2x}{1+2x}}\, dx$

Similar Qn 42

$$= \int \sqrt{\frac{(1-2x)(1-2x)}{(1+2x)(1-2x)}}\, dx = \int \frac{\sqrt{(1-2x)^2}}{\sqrt{1^2-(2x)^2}}\, dx$$

$$= \int \frac{1-2x}{\sqrt{1-(2x)^2}}\, dx = \int \frac{1}{\sqrt{1-(2x)^2}}\, dx - 2\int \frac{x}{\sqrt{1-4x^2}}\, dx$$

let $u = 1-4x^2$

$$= \sin^{-1}\left(\frac{2x}{1}\right) - 2\int \frac{\cancel{x}}{\sqrt{u}} \, \frac{du}{-8\cancel{x}}$$

$$= \sin^{-1}(2x) + \frac{1}{4}\int u^{-\frac{1}{2}} du$$

$$= \sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$$

⑦⑦

$$\int \frac{e^{3x}(1-e^{-6x})}{e^{3x}(1+e^{-6x})}\,dx$$

Similar to Q44

$$= \int \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}\,dx \quad ; \quad u = e^{3x} + e^{-3x}$$

$$= \int \frac{\cancel{e^{3x} - e^{-3x}}}{u} \cdot \frac{\overset{1}{du}}{\underset{3}{\cancel{3e^{3x} - 3e^{-3x}}}}$$

$$= \frac{1}{3}\ln\left|e^{3x} + e^{-3x}\right| + C$$

⑦⑧

$$(\ln 3x)^2 = (\ln 3x)(\ln 3x)$$

$$\ln(3x)^2 = 2\ln 3x$$

$$\int \ln(3x)^2\,dx = 2\int \ln(3x)\,dx$$

$$= 2\int \overset{\frac{dv}{dx}}{1} \cdot \overset{u}{\ln(3x)}\,dx$$

$u = \ln(3x)$, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{3x} \cdot 3$, $v = x$

$= \frac{1}{x}$

$$= 2\left[x\ln(3x) - \int x \cdot \frac{1}{x}\,dx\right]$$

$$= 2\left[x\ln(3x) - x\right] + C$$

(79) $$\int (\ln 3x)^2 \, dx$$

$u = (\ln 3x)^2 \quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = 2(\ln 3x) \cdot \frac{1}{3x} \cdot 3 \quad , \quad v = x$

$= \frac{2 \ln 3x}{x}$

$$= \int 1 \cdot (\ln 3x)^2 \, dx$$

$$= x(\ln 3x)^2 - \int x \cdot \frac{2 \ln 3x}{x} \, dx$$

$$= x(\ln 3x)^2 - \underline{\int 2 \ln 3x \quad dx}$$

↓ from (Q78)

$$= x(\ln 3x)^2 - \underline{\left[2x \ln(3x) - 2x\right]} + C$$

$$= x(\ln 3x)^2 - 2x \ln(3x) + 2x + C$$

(80) $\int \ln(3-x^2)\,dx$; $u = \ln(3-x^2)$, $\frac{dv}{dx} = 1$

$= \int 1 \cdot \ln(3-x^2)\,dx$

$\frac{du}{dx} = \frac{1}{3-x^2} \cdot (-2x)$, $v = x$

$= -\frac{2x}{3-x^2}$

$= x\ln(3-x^2) + \int x \cdot \frac{2x}{3-x^2}\,dx$

$= x\ln\left|3-x^2\right| + 2\int \frac{x^2}{3-x^2}\,dx$

$$\begin{array}{r|l} & -1 \\ -x^2+3 & x^2 \\ & -(x^2-3) \\ \hline & 3 \end{array}$$

$= x\ln|3-x^2| + 2\int -1 + \frac{3}{3-x^2}\,dx$

$= x\ln|3-x^2| + 2(-x) + 6\int \frac{1}{\sqrt{3}^2 - x^2}\,dx$

$= x\ln|3-x^2) - 2x + 6\left[\frac{1}{2\sqrt{3}} \ln\left(\frac{\sqrt{3}+x}{\sqrt{3}-x}\right)\right] + C$

$= x\ln\left|3-x^2\right| - 2x + \sqrt{3}\ln\left|\frac{\sqrt{3}+x}{\sqrt{3}-x}\right| + c$

(81)

$$\int \frac{\sin x}{\sin 2x}\,dx$$

$$= \int \frac{\sin x}{2\sin x \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\cos x}\,dx$$

$$= \frac{1}{2}\int \sec x\,dx$$

$$= \frac{1}{2}\ln\left|\sec x + \tan x\right| + \underline{\underline{C}}.$$

(82)

$$\int \frac{\sec x \cdot \sec x}{(\sin x + \cos x)\cdot \sec x}\,dx$$

$$= \int \frac{\sec^2 x}{\frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}}\,dx$$

$$= \int \frac{\sec^2 x}{\tan x + 1}\,dx \quad ; \; u = \tan x + 1$$

$$= \int \frac{\cancel{\sec^2 x}}{u}\,\frac{du}{\cancel{\sec^2 x}}$$

$$= \ln\left|\tan x + 1\right| + \underline{\underline{C}}$$

⑧③ $\int \ln(x^2+2x+2)\,dx$ $u=\ln(x^2+2x+2)$ $\frac{dv}{dx}=1$

$\frac{du}{dx}=\frac{1}{x^2+2x+2}\cdot(2x+2)$, $v=x$

$=\int 1\cdot\ln(x^2+2x+2)\,dx$

$= x\ln|x^2+2x+2| - \int\frac{(2x+2)x}{x^2+2x+2}\,dx$

$= x\ln|x^2+2x+2| - 2\int\frac{x^2+x}{x^2+2x+2}\,dx$

$$\begin{array}{r} 1 \\ x^2+2x+2\,\overline{)\,x^2+x\qquad} \\ -(x^2+2x+2) \\ \hline -x-2 \end{array}$$

$= x\ln|x^2+2x+2| - 2\int 1 - \frac{x+2}{x^2+2x+2}\,dx.$

$= x\ln|x^2+2x+2| - 2x + 2\int\frac{(x+1)+1}{x^2+2x+2}\,dx$

$= x\ln|x^2+2x+2| - 2x + \int\frac{2x+2}{x^2+2x+2}\,dx + 2\int\frac{1}{(x+1)^2+1^2}\,dx$

$= x\ln|x^2+2x+2| - 2x + \ln|x^2+2x+2| + 2\tan^{-1}(x+1) + C$

(84) $$\int \frac{x^4+2x+2}{x^5+x^4}\,dx$$

$$= \int \frac{x^4+2(x+1)}{x^4(x+1)}\,dx$$

$$= \int \frac{\cancel{x^4}}{\cancel{x^4}(x+1)} + \frac{2\cancel{(x+1)}}{x^4\cancel{(x+1)}}\,dx$$

$$= \int \frac{1}{x+1} + \frac{2}{x^4}\,dx \quad ; \quad 2\int x^{-4}\,dx = \frac{2x^{-3}}{-3}$$

$$= \ln|x+1| - \frac{2}{3x^3} + C$$

$x^2 = 2\sec u \Rightarrow x = \sqrt{2\sec u}$

Diff w.r.t x.

$2x = 2\sec u \tan u \frac{du}{dx}$

$dx = \frac{2\sec u \tan u}{2x} du.$

(85)

$$\int \frac{1}{x\sqrt{x^4-4}}\, dx$$

$$= \int \frac{1}{x\sqrt{(2\sec u)^2-4}} \cdot \frac{\sec u \tan u}{x}\, du$$

$\tan^2 A + 1 = \sec^2 A$

$$= \int \frac{\sec u \ \tan u}{x^2\sqrt{4(\sec^2 u - 1)}}\, du$$

$\sec^2 A - 1 = \tan^2 A$

$$= \int \frac{\cancel{\sec u}\ \cancel{\tan u}}{2\cancel{\sec u}\cdot 2\sqrt{\cancel{\tan^2 u}}}\, du$$

$$= \frac{1}{4}\int 1\, du = \frac{1}{4}u + c$$

; $x^2 = \frac{2}{\cos u}$

$\cos u = \frac{2}{x^2}$

$u = \cos^{-1}\left(\frac{2}{x^2}\right)$

$$= \frac{1}{4}\cos^{-1}\left(\frac{2}{x^2}\right) + C$$

⑧⑥

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx \quad ; \quad x = \sin u$$

$$\frac{dx}{du} = \cos u \Rightarrow dx = \cos u \, du$$

$$= \int \frac{(\sin u)^3}{\sqrt{1-(\sin u)^2}} \cos u \, du \quad ; \quad 1-\sin^2 u = \cos^2 u$$

$$= \int \frac{\sin^3 u \quad \cos u}{\sqrt{\cos^2 u}} \, du$$

$$= \int \sin^3 u \quad du \quad ; \quad \nu = \cos u$$

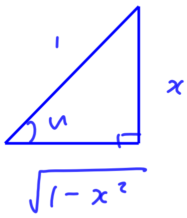
$$= \int \sin^3 u \quad \frac{d\nu}{-\sin u} = -\int \sin^2 u \; d\nu$$

$$= -\int 1 - \cos^2 u \quad d\nu = -\int 1 - \nu^2 \; d\nu$$

$$= -\left[\nu - \frac{\nu^3}{3}\right] + c \quad ;$$

$$= -\left[\cos u - \frac{\cos^3 u}{3}\right] + c$$

$$x = \sin u = \frac{x}{1}$$



$$\cos u = \frac{\sqrt{1-x^2}}{1}$$

$$= -\left[\sqrt{1-x^2} - \frac{\sqrt{1-x^2}^{\,3}}{3}\right] + c$$

\#

$$= -\frac{\sqrt{1-x^2}}{3}\left[3 - \sqrt{1-x^2}^{\,2}\right] + c$$

$$= -\frac{\sqrt{1-x^2}}{3}\left(3 - (1-x^2)\right) + c$$

$$= -\frac{1}{3}\sqrt{1-x^2}\,(2+x^2) + c$$

(87) $$\int \frac{\sin^2 x + 1}{\cos^2 x}\, dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x} + \frac{1}{\cos^2 x}\, dx$$

$$= \int \tan^2 x + \sec^2 x \, dx \;; \quad \overset{\otimes}{\underline{\tan^2 A}} + \overset{\checkmark}{1} = \overset{\checkmark}{\sec^2 A}$$

$$= \int \sec^2 x - 1 + \sec^2 x \, dx$$

$$= \int 2\sec^2 x - 1 \, dx$$

$$= \underline{\underline{2\tan x - x + C}}$$

⑧⑧ $$\int \frac{x^2}{x^2-3}\,dx \qquad \begin{array}{r|l} & \;1 \\ \hline x^2-3 & x^2 \\ & -(x^2-3) \\ \hline & \;\;3 \end{array}$$

$$= \int 1 + \frac{3}{x^2-3}\,dx$$

$$= x + 3\int \frac{1}{x^2-\sqrt{3}^2}\,dx$$

$$= x + 3\left(\frac{1}{2\sqrt{3}}\right)\ln\left(\frac{x-\sqrt{3}}{x+\sqrt{3}}\right) + c.$$

$$= x + \frac{3}{2\sqrt{3}}\ln\left(\frac{x-\sqrt{3}}{x+\sqrt{3}}\right) + C$$

⑧⑨ ⓐ $$\int \frac{x^3-3}{x^2-3}\,dx \qquad \begin{array}{r|l} & \;x \\ \hline x^2-3 & x^3-3 \\ & -(x^3-3x) \\ \hline & \;3x-3 \end{array}$$

$$= \int x + \frac{3x-3}{x^2-3}\,dx$$

$$= \frac{x^2}{2} + \int \frac{3x}{x^2-3} - \frac{3}{x^2-3}\,dx \;;\; \text{let } u = x^2-3$$

$$= \frac{x^2}{2} + \int \frac{3x}{u}\,\frac{du}{2x} - 3\int \frac{1}{x^2-\sqrt{3}^2}\,dx$$

$$= \frac{1}{2}x^2 + \frac{3}{2}\ln\left|x^2-3\right| - 3\left(\frac{1}{2\sqrt{3}}\right)\ln\left(\frac{x-\sqrt{3}}{x+\sqrt{3}}\right) + C$$

$$= \frac{1}{2}x^2 + \frac{3}{2}\ln\left|x^2-3\right| - \frac{3}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C$$

⑧⑨ Ⓑ $\int \frac{x^3}{x^2-3}\,dx$

$$\begin{array}{r|l} & \quad x \\ \hline x^2-3 & x^3 \\ & -(x^3-3x) \\ \hline & \qquad 3x \end{array}$$

$$= \int x + \frac{3x}{x^2-3}\,dx$$

$$= \int x\,dx + \int \frac{3x}{x^2-3}\,dx \quad ; \text{ let } u = x^2-3$$

$$= \frac{x^2}{2} + \int \frac{3x}{u}\,\frac{du}{2x}$$

$$= \frac{1}{2}x^2 + \frac{3}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}x^2 + \frac{3}{2}\ln\left|x^2-3\right| + C$$

(90) $$\int \frac{x^4}{x^2-3}\,dx$$

$$\begin{array}{r|l} & x^2+3 \\ \hline x^2-3 & x^4 \\ & -(x^4-3x^2) \\ \hline & 3x^2 \\ & -(3x^2-9) \\ \hline & 9 \end{array}$$

$$= \int x^2+3 + \frac{9}{x^2-3}\,dx$$

$$= \frac{x^3}{3} + 3x + 9\int \frac{1}{x^2-\sqrt{3}^2}\,dx$$

$$= \frac{1}{3}x^3 + 3x + 9\left(\frac{1}{2\sqrt{3}}\right)\ln\left(\frac{x-\sqrt{3}}{x+\sqrt{3}}\right) + C$$

$$= \frac{1}{3}x^3 + 3x + \frac{9}{2\sqrt{3}}\ln\left(\frac{x-\sqrt{3}}{x+\sqrt{3}}\right) + C$$

㊾2

$$\int \frac{1}{x^3+2x^2+2x+1}\,dx$$

$$\begin{array}{r} x^2+x+1 \\ x+1\overline{)\,x^3+2x^2+2x+1} \\ -(x^3+x^2) \\ \hline x^2+2x \\ -(x^2+x) \\ \hline x+1 \\ \underline{x+1} \end{array}$$

$$= \int \frac{1}{(x+1)(x^2+x+1)}\,dx$$

$$A(x^2+x+1)+(Bx+C)(x+1)=1$$

$$= \int \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}\,dx$$

$x=-1$: $A=1$

constant : $A+C=1$

$C=1-A=0$

x^2: $A+B=0$

$B=-A=-1$

$$= \int \frac{1}{x+1}\,dx + \int \frac{-x}{x^2+x+1}\,dx$$

$$= \ln|x+1| + \int \frac{P(2x+1)+Q}{x^2+x+1}\,dx$$

$P(2x+1)+Q=-x$

$$= \ln|x+1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2-(\frac{1}{2})^2+1}\,dx$$

$P=-1$, $P+Q=0$

$P=-\frac{1}{2}$, $Q=-P=\frac{1}{2}$

$$= \ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\sqrt{\frac{3}{4}}^2}\,dx$$

$$= \ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{1}{2}\left(\frac{1}{\frac{\sqrt{3}}{2}}\right)\tan^{-1}\left(\frac{2(x+\frac{1}{2})}{2(\frac{\sqrt{3}}{2})}\right)+C$$

$$= \ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

A.

(93) $\int \sqrt{\cos x + 1}\ dx$; $\cos 2A = 2\cos^2 A - 1$

$A = \frac{x}{2}$

$$= \int \sqrt{\left(2\cos^2 \frac{x}{2} - 1\right) + 1}\ dx$$

$$= \int \sqrt{2\cos^2 \frac{x}{2}}\ dx$$

$$= \sqrt{2} \int \cos \frac{x}{2}\ dx$$

$$= \sqrt{2}\left(\frac{\sin \frac{x}{2}}{\frac{1}{2}}\right) + C$$

$$= 2\sqrt{2}\ \sin\left(\frac{x}{2}\right) + C$$

//.

(94) $\int \sqrt{\sin x + 1}\ dx$; $\sin 2A = 2\cos A \sin A$

$$= \int \sqrt{2\sin \frac{x}{2} \cos \frac{x}{2} + \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right)}\ dx$$

$$= \int \sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2}\ dx$$

$$= \int \cos \frac{x}{2} + \sin \frac{x}{2}\ dx$$

$$= 2\sin \frac{x}{2} - 2\cos \frac{x}{2} + C$$

//.

(95) $\displaystyle\int \frac{\sec^2 x}{1-\tan^2 x}\,dx$; let $u=\tan x$

$$= \int \frac{\cancel{\sec^2 x}\;1}{1-\;u^2}\;\frac{du}{\cancel{\sec^2 x}\;1}$$

$$= \int \frac{1}{1-u^2}\,du$$

$$= \frac{1}{2(1)}\ln\left(\frac{1+u}{1-u}\right)+C$$

$$= \frac{1}{2}\ln\left(\frac{1+\tan x}{1-\tan x}\right)+C$$

(96)

$$\int \frac{\sin x}{\cos x + \sin x}\, dx$$

$\sin x = A(\cos x + \sin x) + B(-\sin x + \cos x)$

$\left.\begin{array}{l} A - B = 1 \\ A + B = 0 \end{array}\right\}$ $A = +\frac{1}{2}$, $B = -\frac{1}{2}$

$$= \int \frac{A(\cos x + \sin x) + B(-\sin x + \cos x)}{\cos x + \sin x}\, dx$$

$u = \cos x + \sin x$

$$= \frac{1}{2}\int \frac{\cancel{\cos x + \sin x}}{\cancel{\cos x + \sin x}}\, dx - \frac{1}{2}\int \frac{-\sin x + \cos x}{\cos x + \sin x}\, dx \; ; \text{ let } u = \cos x + \sin x$$

$$= \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \frac{-\sin x + \cos x}{u} \frac{du}{-\sin x + \cos x}$$

$$= \frac{1}{2} x - \frac{1}{2} \ln|\cos x + \sin x| + c$$

(97) $\overbrace{\int 1\cdot \sin(\ln x)\,dx}^{I}$;

$u = \sin(\ln x)$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{\cos(\ln x)}{x}$ $\quad v = x$

$$= x\sin(\ln x) - \int x \cdot \frac{\cos(\ln x)}{x}\,dx$$

$$= x\sin(\ln x) - \int 1\cdot\cos(\ln x)\,dx$$

$u_1 = \cos(\ln x)$, $\frac{dv}{dx} = 1$

$\frac{du_1}{dx} = -\frac{\sin(\ln x)}{x}$, $v = x$

$$= x\sin(\ln x) - \left[x\cos(\ln x) + \underbrace{\int \sin(\ln x)\,dx}_{I}\right]$$

$$I = x\sin(\ln x) - x\cos(\ln x) - I$$

$$2I = x\left[\sin(\ln x) - \cos(\ln x)\right] + C$$

$$\therefore \int \sin(\ln x)\,dx = \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$$

(98) $$\int \frac{e^{\tan x}}{1-\sin^2 x}\,dx$$

$$= \int \frac{e^{\tan x}}{\cos^2 x}\,dx$$

$$= \int \sec^2 x \; e^{\tan x}\,dx \quad ; \; u = \tan x$$

$$= \int \sec^2 x \; e^{u} \frac{du}{\sec^2 x} = \int e^{u}\,du$$

$$= e^{\tan x} + C$$ ✓

(99) $$\int \frac{\cot x}{1-\cos 2x}\,dx \quad ; \quad \cos 2A = 1 - 2\sin^2 A$$

$$= \int \frac{\frac{\cos x}{\sin x}}{1-(1-2\sin^2 x)}\,dx$$

$$= \int \frac{\frac{\cos x}{\sin x}}{2\sin^2 x}\,dx$$

$$= \frac{1}{2}\int \frac{\cos x}{(\sin x)^3}\,dx \quad ; \; \text{let } u = \sin x$$

$$= \frac{1}{2}\int \frac{\cos x}{u^3} \frac{du}{\cos x}$$

$$= \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C$$

$$= -\frac{1}{4}(\sin x)^{-2} + C = -\frac{1}{4}\cdot\frac{1}{\sin^2 x} + C$$

$$= -\frac{1}{4}\operatorname{cosec}^2 x \quad + C$$

(100) $\displaystyle\int \frac{e^{-2x}}{e^{-x}+1}\,dx$; let $u = e^{-x}$

$$= \int \frac{(e^{-x})^2}{(e^{-x})+1}\,\frac{du}{-e^{-x}}$$

$$= -\int \frac{u^2}{u+1}\,\frac{du}{u}$$

$$= -\int \frac{u}{u+1}\,du$$

$$\begin{array}{r|l} & \quad 1 \\ u+1 & u \\ & -(u+1) \\ \hline & \quad -1 \end{array}$$

$$= -\int 1 - \frac{1}{u+1}\,du$$

$$= -\left[u - \ln|u+1|\right] + C$$

$$= -\left[e^{-x} - \ln|e^{-x}+1|\right] + C$$

$$= \ln(e^{-x}+1) - e^{-x} + C$$